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The orbifolds of $N = 2$ superconformal theories with $c = 3$

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Abstract. We construct $\mathbb{Z}_M$, $M = 2, 3, 4, 6$ orbifold models of the $N = 2$ superconformal field theories with central charge $c = 3$. Then we check the description of the $\mathbb{Z}_3$, $\mathbb{Z}_4$ and $\mathbb{Z}_6$ orbifolds by the $N = 2$ superconformal Landau–Ginzburg models with $c = 3$, by comparing the spectrum of chiral fields, in particular the Witten index $\mathrm{Tr}(-1)^F$.

1. Introduction

The complete understanding of the modulus space of $N = 2$ superconformal field theories with central charge $c = 3$ needs a description of all its orbifold theories. In a nonlinear σ-model description, this concerns two-dimensional tori and their orbifolds. For $\mathbb{Z}_3$, $\mathbb{Z}_4$ and $\mathbb{Z}_6$ orbifolds, Vafa and Warner [16] made predictions for (chiral, chiral) and (antichiral, antichiral) fields based on Landau–Ginzburg descriptions. Apparently, they never had been checked explicitly. The moduli spaces of those orbifold theories were obtained in [10]. For Landau–Ginzburg descriptions for the three orbifolds, we use the superpotentials $\Phi_1^3 + \Phi_2^3 + \Phi_3^3 + 6a\Phi_1\Phi_2\Phi_3$, $\Phi_1^4 + \Phi_2^4 + a\Phi_1^2\Phi_2^2$ and $\Phi_1^3 + \Phi_2^6 + a\Phi_1^2\Phi_2^2$, respectively. Note that we are interested in one-dimensional modulus spaces, such that one needs superpotentials with one free parameter a or, in other words, singularities of modality one. Correlation functions for these potentials have been studied in [7, 12]. Here we calculate the $\mathbb{Z}_M$ orbifold partition functions and check the predictions of Vafa and Warner. For $c = 6$ similar calculations have been formulated by Eguchi *et al* [5]. There, charges behave in a simpler way than for $c = 3$. When fermions are omitted from the $c = 3$ superconformal theories, one obtains $c = 2$ bosonic theories. In this case the partition function for the $\mathbb{Z}_2$ orbifold was given in [9].

The $N = 2$ superconformal field theories with $c = 3$ [1] are described by a free chiral scalar superfield containing two real bosons or a single complex left (right) boson $\varphi^{\pm}(z) = \varphi^1(z) \pm \mathrm{i}\varphi^2(z)$ ($\bar\varphi^{\pm}(\bar z) = \bar\varphi^1(\bar z) \pm \mathrm{i}\bar\varphi^2(\bar z)$) (each of $c = 1$) and two Majorana–Weyl (MW) fermions or a free complex left (right) fermion $\psi^{\pm}(z) = \psi^1(z) \pm \mathrm{i}\psi^2(z)$ ($\bar\psi^{\pm}(\bar z) = \bar\psi^1(\bar z) \pm \mathrm{i}\bar\psi^2(\bar z)$) (each of $c = \frac{1}{2}$). The action for this system may be written as

$$S = \frac{1}{2\pi}\int \mathrm{d}^2z\,(G_{ij}\partial\varphi^i\bar\partial\varphi^j + B_{ij}\partial\varphi^i\bar\partial\varphi^j + \psi^-\bar\partial\psi^+ + \psi^+\bar\partial\psi^-). \qquad (1)$$

In string theory language, this action corresponds to the superstring compactification on a two-dimensional torus $T^2 = \mathbb{R}^2/\Lambda$. For the two-dimensional lattice Λ, we use a basis $\{e_i\} \in \mathbb{R}$ ($i = 1, 2$). The action (1) depends on four real parameters or moduli, the constant symmetric metric $G_{ij} = \frac{1}{2}e_ie_j$ on T^2, and the antisymmetric tensor field $B_{ij} = -B_{ji}$. It has $N = 2$ superconformal symmetry. Directly from the action, we can determine the generators of the $N = 2$ superconformal algebra, the stress-energy tensor $T(z)$, its super partners $Q^i(z) = Q^1(z) \pm \mathrm{i}Q^2(z)$ $(i = 1, 2)$ and the $U(1)$ current $J(z)$ with conformal dimensions h equal to 2, $\frac{3}{2}$ and 1, respectively:

$$\begin{aligned} T(z) &= -\tfrac{1}{2}\partial\varphi^-(z)\partial\varphi^+(z) - \tfrac{1}{4}\psi^-\partial\psi^+(z) - \tfrac{1}{4}\psi^+(z)\partial\psi^-(z) \\ Q^\pm(z) &= \psi^\mp(z)\partial\varphi^\pm(z) \qquad J(z) = \frac{1}{2}\psi^-(z)\psi^+(z) = \frac{\mathrm{i}}{2}\varepsilon^{ij}\psi^i(z)\psi^j(z). \end{aligned} \tag{2}$$

Similar relations hold for the antiholomorphic (right moving) generators of the $N = 2$ superconformal algebra. They have the Laurent expansions

$$T(z) = \sum_{n=-\infty}^{+\infty} L_n z^{-n-2} \qquad Q^i(z) = \sum_{r=-\infty}^{+\infty} Q_r z^{-r-\frac{3}{2}} \qquad J(z) = \sum_{n=-\infty}^{+\infty} J_n z^{-n-1}$$

and satisfy the $N = 2$ superconformal algebra that can be found in [1, 14]. There are three different $N = 2$ superconformal algebras, namely Ramond (R) (or periodic (P)), Neveu–Schwarz (NS) (or antiperiodic (A)) and twisted (T) algebras which correspond to different ways of choosing boundary conditions on the cylinder. Whatever boundary condition we choose, the Virasoro generator L_n is always integrally moded, because the bosonic stress-energy tensor is always periodic on the cylinder. For the R algebra, J_n and Q_r^i are integrally moded, i.e. n and r run over integral values. For the NS algebra, J_n are integrally moded, Q_r^i are half-integrally moded, i.e. r run over half-integral values. The T algebra has integer modes for Q_r^1 and half integer modes for J_n and Q_r^2.

A field satisfying $h = \pm q/2$ is a left chiral or left antichiral primary field. (Similarly, a field satisfying $\bar{h} = \pm\bar{q}/2$ is a right chiral or right antichiral primary field.) Note that the fermionic fields $\{\psi^\pm(z), \bar{\psi}^\pm(\bar{z})\}$ all satisfy the above condition since they have charge ± 1 and conformal dimension $\frac{1}{2}$ for both the left movers and right movers. The left primary chiral fields are $\{1, \psi^+(z)\}$; the right chiral primary fields are $\{1, \bar{\psi}^+(\bar{z})\}$. The left and right antichiral primary fields are obtained from these by complex conjugation. Note that the conformal dimensions and $U(1)$ charges of a unique highest left–right chiral or antichiral primary field are $(h, \bar{h}) = (c/6, c/6) = (\frac{1}{2}, \frac{1}{2})$ and $(q, \bar{q}) = (\pm c/3, \pm c/3) = (\pm 1, \pm 1)$, respectively (here $c = 3$).

In general for $N = 2$ superconformal theories, there are four types of ring [11] arising from the various combinations of left–right chiral and left–right antichiral fields. We denote these rings by (c, c), (a, a), (a, c), (c, a). They are pairwise conjugate. For the $\mathbb{Z}_M$, $M \in \{3, 4, 6\}$, orbifolds of $N = 2$ superconformal theories with $c = 3$, and for $N = 2$ superconformal Landau–Ginzburg models, one obtains only (c, c) and its conjugate (a, a) rings. For such models, the (a, c) and (c, a) rings are trivial and consist only of the identity operator. We shall see this point explicitly in the discussion of $\mathbb{Z}_M$ orbifolds and Landau–Ginzburg models.

The basic linearly independent elements of the (c, c) ring of the $N = 2$ superconformal field theory with $c = 3$ are given by

$$\mathcal{R}_{(c,c)} = \{1, \psi^+(z), \bar{\psi}^+(\bar{z}), \psi^+(z)\bar{\psi}^+(\bar{z})\}. \tag{3}$$

Similarly, for the (a, c) ring one has

$$\mathcal{R}_{(a,c)} = \{1, \psi^-(z), \bar{\psi}^+(\bar{z}), \psi^-(z)\bar{\psi}^+(\bar{z})\}. \tag{4}$$

The elements of the two other rings $\mathcal{R}_{(a,a)}$ and $\mathcal{R}_{(c,a)}$ are obtained from $\mathcal{R}_{(c,c)}$ and $\mathcal{R}_{(a,c)}$ by complex conjugation.

The conformal dimensions and $U(1)$ charges of the ground states of the R sector are $(h, \bar{h}) = (c/24, c/24) = (\frac{1}{8}, \frac{1}{8})$ and $(q, \bar{q}) = (\pm\frac{1}{2}, \pm\frac{1}{2})$, which also contribute to the Witten index $\mathrm{Tr}(-1)^F$ [17]. The operator $(-1)^F$, where $F = F_{\mathrm{L}} + F_{\mathrm{R}}$, and F_{L}, F_{R} are left–right moving fermion numbers, defined to anticommute with all the fermionic operators $(-1)^F\psi(z) =$

$-\psi(z)(-1)^F$, and to commute with all the bosonic operators $(-1)^F\varphi(z) = \varphi(z)(-1)^F$, as well as to satisfy $((-1)^F)^2 = 1$. It can be defined in terms of zero-mode $U(1)$ current as

$$(-1)^F = \mathrm{e}^{\pi\mathrm{i}(J_0-\bar{J}_0)}.$$

It is well known that one can connect the NS sector to the R sector by the spectral flow [14] operation. It is a continuous transformation and has the following form:

$$\begin{aligned} L_n^\eta &= L_n + \eta J_n + \frac{c}{6}\eta^2\delta_{n,0} \\ J_n^\eta &= J_n + \frac{c}{3}\eta\delta_{n,0} \\ Q_r^{\pm\eta} &= Q_{r\pm\eta}^{\pm}. \end{aligned}$$

The η twisted operators L_n^η, $Q_r^{\pm\eta}$ and J_n^η still satisfy the $N = 2$ superconformal algebra for an arbitrary value of the parameter η. In particular, the zero-mode eigenvalues h of L_0 and q of J_0 are changed by spectral flow as

$$h_\eta = h + \eta q + \eta^2\frac{c}{6} \qquad q_\eta = q + \eta\frac{c}{3}. \tag{5}$$

By (5) with flow parameter $\eta = \frac{1}{2}$, the ground states of the R sector with conformal dimension $(h, \bar{h}) = (\frac{1}{8}, \frac{1}{8})$ and charge $(q, \bar{q}) = (\pm\frac{1}{2}, \pm\frac{1}{2})$ flow to the NS chiral primary fields with conformal dimension $(h, \bar{h}) = (\frac{1}{2}, \frac{1}{2})$ and charge $(q, \bar{q}) = (+1, +1)$, or $(h, \bar{h}) = (q, \bar{q}) = (0, 0)$. The flow between the NS and NS as well as R and R can be obtained by the flow parameter $\eta = 1$. Besides, under the left–right symmetric spectral flow, $q - \bar{q} \in \mathbb{Z}$ does not change. Thus the Witten index [10] is

$$\begin{aligned} \mathrm{Tr}(-1)^F &= \mathrm{Tr}_{\mathrm{R}}[(-1)^{J_0-\bar{J}_0} q^{L_0-\frac{c}{24}} \bar{q}^{\bar{L}_0-\frac{c}{24}}] \\ &= \mathrm{Tr}_{\mathcal{H}_\eta}[(-1)^{J_0^\eta-\bar{J}_0^\eta} q^{L_0^\eta-\frac{c}{24}} \bar{q}^{\bar{L}_0^\eta-\frac{c}{24}}] \\ &= \mathrm{Tr}_{\mathrm{NS}}[(-1)^{J_0-\bar{J}_0} q^{L_0+\frac{1}{2}J_0} \bar{q}^{\bar{L}_0+\frac{1}{2}\bar{J}_0}] = \sum_{\mathcal{R}} \mathrm{e}^{\mathrm{i}\pi(q-\bar{q})} \end{aligned} \tag{6}$$

where the $\mathcal{H}_\eta$ in the second line is the Hilbert space of states which is twisted by the parameter η. The $\mathcal{R}$ in the last line denotes the chiral ring. The first line implies that the ground state of the R sector gives nonvanishing contribution to the Witten index. The second line is obtained by applying the spectral flow to the first line. By setting $\eta = \frac{1}{2}$ one can flow from the R sector to the NS sector. (Note that $J_0^\eta - \bar{J}_0^\eta = J_0 - \bar{J}_0$.) Thus the Witten index receives contributions from either the ground states of the R sector or the chiral primary states of NS sector. The only difference between the charges of the NS chiral primary states and that of the R ground states is $c/6$.

The Poincaré polynomial [11] is

$$P(t, \bar{t}) = \mathrm{Tr}_{\mathcal{R}}\, t^{J_0\bar{J}_0} \tag{7}$$

which satisfies a duality relation $P(t, \bar{t}) = (t\bar{t})^{c/3} P(1/t, 1/\bar{t})$. Here t and $\bar{t}$ can be regarded as independent variables. By (3), (6) and (7), the Witten index and the Poincaré polynomial are

$$\mathrm{Tr}(-1)^F = 0 \qquad P(t, \bar{t})_{(c,c)} = 1 + t + \bar{t} + t\bar{t}. \tag{8}$$

One notes that the Poincaré polynomial (8) and ring structure for (c, c) and (a, c) primary fields are isomorphic. However, this is not true in general.

The partition function for the $N = 2$ superconformal theories with $c = 3$ is constructed by tensoring the theory of a complex free boson defined on a two-dimensional torus T^2 in the presence of constant background fields, with the theory of a single complex Dirac fermion, namely

$$Z(\tau, \rho, z) := Z(\tau, \rho, \sigma) Z_{\mathrm{Dirac}}(\sigma, z).$$

In the following we briefly discuss how the explicit expression of $Z(\tau, \rho, z)$ can be formulated. The $Z(\tau, \rho, \sigma)$ is the modular invariant partition function for two real bosons compactified on the two-dimensional torus [6]

$$Z(\tau, \rho) := Z(\tau, \rho, \sigma) = \operatorname{tr} q^{L_0^b - \frac{1}{12}} \bar{q}^{\bar{L}_0^b - \frac{1}{12}} = \frac{1}{|\eta^2(\sigma)|^2} \sum_{\substack{n_1, m_1 \\ n_2, m_2}} q^{\frac{p^2}{2}} \bar{q}^{\frac{\bar{p}^2}{2}} \tag{9}$$

where $q = \mathrm{e}^{2\pi \mathrm{i}\sigma}$, $\sigma = \sigma_1 + \mathrm{i}\sigma_2$ parametrizes the world sheet torus, and $\eta(\sigma)$ is the Dedekind eta function defined as

$$\eta(\sigma) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n).$$

The Virasoro zero-mode operators for the bosons in (9) are given by

$$L_0^b = \sum_{n>0} \alpha_{-n}^i \alpha_n^i + \tfrac{1}{2} p^2 \qquad \bar{L}_0^b = \sum_{n>0} \bar{\alpha}_{-n}^i \bar{\alpha}_n^i + \tfrac{1}{2} \bar{p}^2. \tag{10}$$

The left–right moving zero-mode momentum p and $\bar{p}$ in (9) are defined as

$$(p, \bar{p}) := (n_i e^{*i} + e^{*i} B_{ji} m^j + \tfrac{1}{2} e_j m^j, n_i e^{*i} + e^{*i} B_{ji} m^j - \tfrac{1}{2} e_j m^j) \tag{11}$$

where $\{e_i^*\}$ are basis vectors for the dual lattice Λ^* of Λ, which satisfies $e_i e_j^* = \delta_{ij}$ such that $e^{*i} e^{*j} = \frac{1}{2} G^{ij}$; the integers n_i and m_i are the momentum and winding numbers. The action of L_0^b and $\bar{L}_0^b$ in (10) on the ground state $|m_1, m_2, n_1, n_2\rangle$, which is labelled by the momentum and winding numbers, is given by

$$L_0^b |m_1, m_2, n_1, n_2\rangle = \tfrac{1}{2} p^2 |m_1, m_2, n_1, n_2\rangle \qquad \bar{L}_0^b |m_1, m_2, n_1, n_2\rangle = \tfrac{1}{2} \bar{p}^2 |m_1, m_2, n_1, n_2\rangle$$

where we have used $\alpha_n^i |m_1, m_2, n_1, n_2\rangle = 0$ and $\bar{\alpha}_m^j |m_1, m_2, n_1, n_2\rangle = 0$ for $n > 0, m > 0$. It is well known [13] that the momenta in (11) form a four-dimensional Lorentzian lattice with scalar product $(p, \bar{p}) \cdot (p', \bar{p}') = (p \cdot p' - \bar{p} \cdot \bar{p}')$, which is even (because $p^2 - \bar{p}^2 = 2m^i n_i \in 2\mathbb{Z}$) and self-dual (because $\Lambda = \Lambda^*$). From (11), we easily write

$$p^2(\bar{p}^2) = \tfrac{1}{2} n_i n_j G^{ij} + n_i m_j B_{jl} G^{il} \pm n_i m_i + \tfrac{1}{2} m_i m_j (G_{ij} + B_{jk} B_{il} G^{kl}). \tag{12}$$

In the two-dimensional case, it is convenient to group the four real parameters (G_{11}, G_{12}, G_{22} and B_{12}) in terms of two parameters τ and ρ in the upper complex half-plane as follows:

$$\tau = \tau_1 + \mathrm{i}\tau_2 = \frac{G_{12}}{G_{22}} + \mathrm{i}\frac{\sqrt{G}}{G_{22}} \qquad \rho = \rho_1 + \mathrm{i}\rho_2 = B_{12} + \mathrm{i}\sqrt{G}.$$

Here τ represents the complex structure of the target space torus T^2, and ρ is its complexified Kähler structure; both take values on the complex upper half-plane; $G = \det(G_{ij})$. Now we write (12) in terms of τ and ρ in the following form:

$$p^2 = \frac{1}{2\tau_2\rho_2} |n_1 - \tau n_2 - \rho(m_2 + \tau m_1)|^2$$

$$\bar{p}^2 = \frac{1}{2\tau_2\rho_2} |n_1 - \tau n_2 - \bar{\rho}(m_2 + \tau m_1)|^2.$$

Finally, the torus partition function (9) takes the form

$$Z(\tau, \rho) = \frac{1}{|\eta^2(\sigma)|^2} \sum_{\substack{n_1, m_1 \\ n_2, m_2}} q^{\frac{1}{4\tau_2\rho_2} |n_1 - \tau n_2 - \rho(m_2 + \tau m_1)|^2} \bar{q}^{\frac{1}{4\tau_2\rho_2} |n_1 - \tau n_2 - \bar{\rho}(m_2 + \tau m_1)|^2}. \tag{13}$$

If $\tau_1 = \rho_1 = 0$ (or $G_{12} = B_{12} = 0$), then the torus partition function (13) is the product of two circle partition functions [8] at $c = 1$ with radius $r_1 = \sqrt{G_{22}} = \sqrt{\rho_2/\tau_2}$ and $r_2 = \sqrt{G_{11}} = \sqrt{\tau_2\rho_2}$

$$Z(\tau_2, \rho_2) = Z^{c=1}\left(\sqrt{\rho_2/\tau_2}\right) Z^{c=1}\left(\sqrt{\tau_2\rho_2}\right).$$

The partition function for the Dirac fermion can be constructed by taking equal spin structures for the left and right fermions [8]

$$\begin{aligned} Z_{\mathrm{Dirac}}(\sigma, z) &= \operatorname{tr} q^{L_0^f - \frac{1}{24}} \bar{q}^{\bar{L}_0^f - \frac{1}{24}} y^{J_0} \bar{y}^{\bar{J}_0} \\ &= \frac{1}{2}\left(\left|\frac{\vartheta_1(z,\sigma)}{\eta(\sigma)}\right|^2 + \left|\frac{\vartheta_2(z,\sigma)}{\eta(\sigma)}\right|^2 + \left|\frac{\vartheta_3(z,\sigma)}{\eta(\sigma)}\right|^2 + \left|\frac{\vartheta_4(z,\sigma)}{\eta(\sigma)}\right|^2\right) \end{aligned} \tag{14}$$

where $y = \mathrm{e}^{2\pi \mathrm{i} z}$. Since the fermionic theory splits into NS and R sectors, the Virasoro zero-mode generator for the Dirac fermions in (14) is given by

$$\begin{aligned} L_0^f &= \sum_{n>0} n d_{-n}^i d_n^i \qquad && n \in \mathbb{Z} + \tfrac{1}{2} \qquad && (\mathrm{NS}) \\ L_0^f &= \sum_{n>0} n d_{-n}^i d_n^i + \tfrac{1}{8} \qquad && n \in \mathbb{Z} \qquad && (\mathrm{R}). \end{aligned}$$

A similar relation is true for the right-moving component. The classical Jacobi theta functions $\vartheta_i(z, \sigma)$, $i \in \{1, 2, 3, 4\}$ in (14) are defined in terms of sums and products as

$$\begin{aligned} \theta_1(z,\sigma) &= -\mathrm{i} \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{1}{2}(n-\frac{1}{2})^2} y^{n-\frac{1}{2}} = -\mathrm{i} y^{\frac{1}{2}} q^{\frac{1}{8}} \prod_{n=1}^{\infty} (1-q^n)(1-yq^n)(1-y^{-1}q^{n-1}) \\ \theta_2(z,\sigma) &= \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}(n-\frac{1}{2})^2} y^{n-\frac{1}{2}} = y^{\frac{1}{2}} q^{\frac{1}{8}} \prod_{n=1}^{\infty} (1-q^n)(1+yq^n)(1+y^{-1}q^{n-1}) \\ \theta_3(z,\sigma) &= \sum_{n=-\infty}^{\infty} q^{\frac{n^2}{2}} y^n = \prod_{n=1}^{\infty} (1-q^n)(1+yq^{n-\frac{1}{2}})(1+y^{-1}q^{n-\frac{1}{2}}) \\ \theta_4(z,\sigma) &= \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n^2}{2}} y^n = \prod_{n=1}^{\infty} (1-q^n)(1-yq^{n-\frac{1}{2}})(1-y^{-1}q^{n-\frac{1}{2}}). \end{aligned}$$

The partition function for the $N = 2$ superconformal theories with $c = 3$ is thus given as

$$\begin{aligned} Z(\tau, \rho, z) &:= Z(\tau, \rho) Z_{\mathrm{Dirac}}(\sigma, z) \\ &= \frac{1}{|\eta^2(\sigma)|^2} q^{\frac{1}{4\tau_2\rho_2}|n_1 - \tau n_2 - \rho(m_2 + \tau m_1)|^2} \bar{q}^{\frac{1}{4\tau_2\rho_2}|n_1 - \tau n_2 - \bar{\rho}(m_2 + \tau m_1)|^2} \\ &\quad \times \frac{1}{2}\left(\left|\frac{\vartheta_1(z,\sigma)}{\eta(\sigma)}\right|^2 + \left|\frac{\vartheta_2(z,\sigma)}{\eta(\sigma)}\right|^2 + \left|\frac{\vartheta_3(z,\sigma)}{\eta(\sigma)}\right|^2 + \left|\frac{\vartheta_4(z,\sigma)}{\eta(\sigma)}\right|^2\right). \end{aligned} \tag{15}$$

2. General prescription for $\mathbb{Z}_M$ orbifold construction

In this section we will give the general procedure for the construction of the $\mathbb{Z}_M$ orbifolds. In fact there are not many two-dimensional $\mathbb{Z}_M$ orbifolds, because the order M rotation must be an automorphism of some two-dimensional lattice; therefore $\mathbb{Z}_M$ must have order $M = 2, 3, 4$ and 6. $M = 3$ and $M = 6$ require the hexagonal lattice ($\tau = \mathrm{e}^{2\pi\mathrm{i}/3}$); $M = 4$ requires a square lattice ($\tau = \mathrm{i}$). Under the $\mathbb{Z}_M$ symmetry bosonic fields and its modes $\alpha_n^{\pm}$ transform as

$$(g^k\varphi)^{\pm}(z) = \mathrm{e}^{\pm\frac{2\pi\mathrm{i}k}{M}} \varphi^{\pm}(z) \qquad g^k \alpha_n^{\pm} g^{-k} = \mathrm{e}^{\pm\frac{2\pi\mathrm{i}k}{M}} \alpha_n^{\pm} \qquad k = 1, 2, \ldots, M-1. \tag{16}$$

Since we want to discuss superconformal orbifold theories, we should include the worldsheet fermion ψs as well. They transform as

$$(g^k\psi)^{\pm}(z) = \mathrm{e}^{\pm\frac{2\pi \mathrm{i}k}{M}}\psi^{\pm}(z) \qquad g^k d_n^{\pm} g^{-k} = \mathrm{e}^{\pm\frac{2\pi \mathrm{i}k}{M}} d_n^{\pm} \qquad k = 1, 2, \ldots, M-1. \tag{17}$$

In fact this is also required by the $N = 2$ superconformal invariance. The $\mathbb{Z}_M$ rotations are the symmetries of both the action (1) and $N = 2$ world sheet supersymmetry generators (2). Thus the two-dimensional $N = 2$ superconformal orbifold models $T^2/\mathbb{Z}_M$ may be constructed by identifying points of the two-dimensional torus T^2 under the symmetry group $\mathbb{Z}_M$.

Let $\tilde{\mathcal{H}}$ be the Hilbert space of an orbifold theory. It has two sectors, namely untwisted and twisted sectors, i.e. $\tilde{\mathcal{H}} = \tilde{\mathcal{H}}_u \oplus \tilde{\mathcal{H}}_t$. Let us consider first the untwisted sector of the orbifold theory. The untwisted Hilbert space will be a subspace of the Hilbert space for the $N = 2$ theories with $c = 3$. In the path integral for the partition function this means that the bosonic fields obey periodic boundary conditions along the space direction of the torus and twisted periodic boundary conditions in time. So on an orbifold, the untwisted sector boundary conditions on the bosonic field are given as

$$\begin{aligned} \varphi^+(1) &= \varphi^+(0) + 2\pi\Lambda \\ \varphi^+(\sigma) &= g\varphi^+(0) + 2\pi\Lambda \end{aligned} \tag{18}$$

where $g \in \mathbb{Z}_M$. For an R or NS fermion one has

$$\begin{aligned} \psi^+(1) &= \pm\psi^+(0) \\ \psi^+(\sigma) &= \pm g\psi^+(0). \end{aligned} \tag{19}$$

Under the above boundary conditions, the bosonic field has the expansion

$$\varphi^+(z) = q^+ - \mathrm{i}p^+ \ln z + \mathrm{i}\sum_{n\neq 0}\frac{1}{n}\alpha_n^+ z^{-n} \tag{20}$$

and for the fermionic field one has

$$\psi^+(z) = \sum_n d_n^+ z^{-n} \begin{cases} n \in \mathbb{Z} & \text{(R)} \\ n \in \mathbb{Z} + \frac{1}{2} & \text{(NS).} \end{cases} \tag{21}$$

The untwisted Hilbert space $\tilde{\mathcal{H}}_u$ decomposes into the $\mathbb{Z}_M$ invariant and noninvariant space of states. In order to construct consistent models, we must project out the group noninvariant space of states. In the Hamiltonian formalism, group invariant states are obtained by insertion of the projection operator $P = \frac{1}{|\mathbb{Z}_M|}\sum_{g\in\mathbb{Z}_M} g$ into the trace over states. Here $|\mathbb{Z}_M|$ is the number of elements in $\mathbb{Z}_M$ and the sum $\sum g$ runs over all elements in $\mathbb{Z}_M$. Thus the untwisted sector partition function is

$$Z_u = \mathrm{tr}_{\tilde{\mathcal{H}}_u}\, P q^{L_0 - \frac{1}{8}} \bar{q}^{\bar{L}_0 - \frac{1}{8}} y^{J_0} \bar{y}^{\bar{J}_0}. \tag{22}$$

Here $\mathrm{tr}_{\tilde{\mathcal{H}}_u}$ denotes the trace in the untwisted Hilbert space sectors and $L_0 = L_0^b + L_0^f$. In the path integral formalism, projection onto group invariant states in the untwisted sector is represented as

$$Z_u = \frac{1}{|\mathbb{Z}_M|}\sum_{g\in\mathbb{Z}_M} g\underset{1}{\boxed{}}$$

where we sum over all possible twistings in the time direction of the torus. $g\underset{1}{\boxed{}}$ represents boundary conditions on any generic fields in the theory twisted by g in the time direction of the torus. The partition function of the original model is simply given by $Z = 1\underset{1}{\boxed{}}$.

The untwisted sector partition function is not modular invariant; one should take into account the contributions of the T sector Hilbert space of states. For each element $h \in \mathbb{Z}_M$ one can construct a twisted Hilbert space $\tilde{\mathcal{H}}_h$. In the path integral description the bosonic field obeys the twisted boundary conditions

$$\begin{aligned} \varphi^+(1) &= h\varphi^+(0) + 2\pi\Lambda \\ \varphi^+(\sigma) &= g\varphi^+(0) + 2\pi\Lambda. \end{aligned} \tag{23}$$

For R or NS fermions one has

$$\begin{aligned} \psi^+(1) &= \pm h\psi^+(0) \\ \psi^+(\sigma) &= \pm g\psi^+(0) \end{aligned} \tag{24}$$

where h and g are twists on the fields in the space and time direction of the torus. The mode expansion of the bosonic field which satisfies the boundary conditions (23) is

$$\varphi^+(z) = q_f^+ + \mathrm{i} \sum_{n\in\mathbb{Z}+k/M} \frac{1}{n}\alpha_n^+ z^{-n}. \tag{25}$$

One cannot have nonzero momentum or winding number here, since they are not consistent with the twisted boundary conditions. In this mode expansion q_f^+ denote the fixed points of T^2 under the $\mathbb{Z}_M$ symmetry. The index f labels these fixed points. The mode expansion of the fermionic field which satisfies the boundary conditions (24) is

$$\psi^+(z) = \sum_{n\in\mathbb{Z}+k/M+\frac{1}{2}-s/2} d_n^+ z^{-n} \qquad k = 1, \ldots, M-1 \tag{26}$$

where s is equal to zero in the NS sector, and to one in the R sector. The twisted Hilbert space $\tilde{\mathcal{H}}_t$ decomposes into $\mathbb{Z}_M$ invariant and noninvariant spaces of states. To construct consistent models, we again have to project onto group invariant states. In the Hamiltonian formalism, group invariant states are obtained by insertion of the projection operator $P_h := \frac{1}{|\mathbb{Z}_M|}\sum_{g\in\mathbb{Z}_M:[g,h]=0} g$ into the trace over states. In the path integral formalism, projection onto group invariant states in the T sector is represented as

$$Z_t = \frac{1}{|\mathbb{Z}_M|} \sum_{\substack{g,h\in\mathbb{Z}_M,\\ h\neq 1,[g,h]=0}} {}_{g}\underset{h}{\square}$$

where ${}_{g}\underset{h}{\square}$ represents boundary conditions on the fields twisted by g and h in the time and space direction of the torus. Thus the T sector partition function has the form

$$Z_t = \sum_{h\in\mathbb{Z}_M,h\neq 1} \mathrm{tr}_{\tilde{\mathcal{H}}_h} P_h q^{L_0-\frac{1}{8}} \bar{q}^{\bar{L}_0-\frac{1}{8}} y^{J_0} \bar{y}^{\bar{J}_0} = \frac{1}{|\mathbb{Z}_M|} \sum_{\substack{g,h\in\mathbb{Z}_M,\\ h\neq 1,[g,h]=0}} {}_{g}\underset{h}{\square}. \tag{27}$$

In fact, one may obtain the T sector partition function from (22) by modular transformations $\sigma \to \sigma + 1$ and $\sigma \to -1/\sigma$. Thus, the total modular invariant $\mathbb{Z}_M$ orbifold partition function is a sum of (22) and (27):

$$\begin{aligned} Z_{\mathbb{Z}_M-\mathrm{orb}} &= \frac{1}{|\mathbb{Z}_M|} \sum_{g\in\mathbb{Z}_M} {}_{g}\underset{1}{\square} + \frac{1}{|\mathbb{Z}_M|} \sum_{g,h\in\mathbb{Z}_M,h\neq 1} {}_{g}\underset{h}{\square} \\ &= \frac{1}{|\mathbb{Z}_M|} \sum_{\substack{g,h\in\mathbb{Z}_M,\\ [g,h]=0}} {}_{g}\underset{h}{\square} = \sum_{h\in\mathbb{Z}_M} \mathrm{tr}_{\tilde{\mathcal{H}}_h} P_h q^{L_0-\frac{1}{8}} \bar{q}^{\bar{L}_0-\frac{1}{8}} y^{J_0} \bar{y}^{\bar{J}_0} \end{aligned} \tag{28}$$

where we set $\tilde{\mathcal{H}}_1 := \tilde{\mathcal{H}}_u$ and $P_1 := P$. There is no discrete torsion for the $\mathbb{Z}_M$ orbifolds, since all boxes ${}_{g}\underset{h}{\square}$ are related by modular tranformations to a box of type ${}_{g}\underset{1}{\square}$. Mathematically, the discrete torsion for a discrete group G is obtained from the cohomology $H_2(G)$, which vanishes for $G = \mathbb{Z}_M$ [15].

In summary, in order to construct an orbifold model, one first formulates the Hilbert space of states on the torus, then one projects onto the group invariant states, finally one includes T sector contributions. For more details see [2–4].

3. The $\mathbb{Z}_2$ orbifold

The two-dimensional $N = 2$ superconformal $\mathbb{Z}_2$ orbifold model $T^2/\mathbb{Z}_2$ can be constructed from (15) for arbitrary τ and ρ. Thus we may now produce another family of theories, i.e. $\mathbb{Z}_2$ orbifold superconformal field theories with the same set of moduli as the $N = 2$ theories with $c = 3$ by following the general orbifold prescription introduced in section 2. The action of $g \in \mathbb{Z}_2$ on the bosonic Hilbert space sectors $|m_1, m_2, n_1, n_2\rangle$ is given by

$$g|m_1, m_2, n_1, n_2\rangle = |-m_1, -m_2, -n_1, -n_2\rangle. \tag{29}$$

In the following, we only discuss the bosonic part since the sum over the spin structures for the Dirac fermion is invariant under $\psi^\pm \to -\psi^\pm$. Under the $\mathbb{Z}_2$ symmetry the untwisted bosonic Hilbert spaces $\tilde{\mathcal{H}}_u$ decomposes into $g = \pm 1$ eigenspaces $\tilde{\mathcal{H}}_u = \tilde{\mathcal{H}}_u^+ \oplus \tilde{\mathcal{H}}_u^-$ as

$$\begin{aligned}
\tilde{\mathcal{H}}_u^+ = &\{\alpha^+_{-k_1} \ldots \alpha^+_{-k_l} \bar{\alpha}^+_{-k_{l+1}} \ldots \bar{\alpha}^+_{-k_{2j}} (1+g)|m_1, m_2, n_1, n_2\rangle\} \\
&+ \{\alpha^+_{-k_1} \ldots \alpha^+_{-k_l} \bar{\alpha}^+_{-k_{l+1}} \ldots \bar{\alpha}^+_{-k_{2j+1}} (1-g)|m_1, m_2, n_1, n_2\rangle\} \\
\tilde{\mathcal{H}}_u^- = &\{\alpha^+_{-k_1} \ldots \alpha^+_{-k_l} \bar{\alpha}^+_{-k_{l+1}} \ldots \bar{\alpha}^+_{-k_{2j+1}} (1+g)|m_1, m_2, n_1, n_2\rangle\} \\
&+ \{\alpha^+_{-k_1} \ldots \alpha^+_{-k_l} \bar{\alpha}^+_{-k_{l+1}} \ldots \bar{\alpha}^+_{-k_{2j}} (1-g)|m_1, m_2, n_1, n_2\rangle\}
\end{aligned}$$

where k_i takes positive integer values. By (22), the untwisted $\mathbb{Z}_2$ orbifold partition function is

$$Z_u = (q\bar{q})^{-\frac{1}{8}} \operatorname{tr}_{\tilde{\mathcal{H}}_u} \tfrac{1}{2}(1+g) q^{L_0} \bar{q}^{\bar{L}_0} y^{J_0} \bar{y}^{\bar{J}_0}.$$

The first term in the trace is equal to the partition function in (15) since there is no twist along the two cycles of the torus. The second term in the trace with g inserted only receives a contribution from the sector $m_1 = m_2 = n_1 = n_2 = 0$ because each state obtained by acting on $(1+g)|m_1, m_2, n_1, n_2\rangle$ with creation operators has a counterpart with the same L_0 eigenvalue obtained by acting on $(1-g)|m_1, m_2, n_1, n_2\rangle$ with the same creation operators; however, these two states have opposite eigenvalues under $g \in \mathbb{Z}_2$, and their contributions cancel in the trace. Thus, only the states obtained by creation operators α^+_{-k} or $\bar{\alpha}^+_{-k}$ acting on the vacuum $|0, 0, 0, 0\rangle$ will contribute. Therefore the overall untwisted sector partition function is

$$\begin{aligned}
Z_u &= \frac{1}{2}\left(\frac{1}{|\eta^2|^2} \sum_{\substack{n_1, m_1\\ n_2, m_2}} q^{\frac{p^2}{2}} \bar{q}^{\frac{\bar{p}^2}{2}} + \frac{(q\bar{q})^{-\frac{1}{12}}}{\prod_{n=1}^{\infty}(1+q^n)^2(1+\bar{q}^n)^2}\right) Z_{\mathrm{Dirac}} \\
&= \frac{1}{2}\left(Z(\tau, \rho) + 4\left|\frac{\eta(\sigma)}{\vartheta_2(\sigma)}\right|^2\right) Z_{\mathrm{Dirac}}.
\end{aligned}$$

Under the symmetry action g: $\varphi^+ \to -\varphi^+$ the torus has four fixed points. This implies that there are four twisted ground states with conformal dimension $h = \bar{h} = \frac{1}{8}$. So one may build four distinct Hilbert space sectors. However, these sectors lead to isomorphic physics, as they are related by translation symmetry of the torus. Denote the four T sector ground states by $|\frac{1}{8}, \frac{1}{8}\rangle_f$, where $f = 1, 2, 3, 4$. Like the untwisted bosonic Hilbert space sector, the twisted bosonic Hilbert space decomposes into $g = \pm 1$ eigenspaces $\tilde{\mathcal{H}}_t = \tilde{\mathcal{H}}_t^+ \oplus \tilde{\mathcal{H}}_t^-$ as

$$\begin{aligned}
\tilde{\mathcal{H}}_t^+ &= \alpha^+_{-k_1} \ldots \alpha^+_{-k_l} \bar{\alpha}^+_{-k_{l+1}} \ldots \bar{\alpha}^+_{-k_{2j}} |\tfrac{1}{8}, \tfrac{1}{8}\rangle_f \\
\tilde{\mathcal{H}}_t^- &= \alpha^+_{-k_1} \ldots \alpha^+_{-k_l} \bar{\alpha}^+_{-k_{l+1}} \ldots \bar{\alpha}^+_{-k_{2j+1}} |\tfrac{1}{8}, \tfrac{1}{8}\rangle_f
\end{aligned}$$

where k_i takes half positive integer values. By (27), the T sector partition function is

$$\begin{aligned} Z_t &= (q\bar{q})^{-\frac{1}{12}} \operatorname{tr}_{\tilde{\mathcal{H}}_t} \tfrac{1}{2}(1+g) q^{L_0} \bar{q}^{\bar{L}_0} Z_{\mathrm{Dirac}} \\ &= 4 \times \frac{1}{2} \left(\left| \frac{q^{\frac{1}{24}}}{\prod_{n=1}^{\infty}(1-q^{n-\frac{1}{2}})^2} \right|^2 + \left| \frac{q^{\frac{1}{24}}}{\prod_{n=1}^{\infty}(1+q^{n-\frac{1}{2}})^2} \right|^2 \right) Z_{\mathrm{Dirac}} \\ &= 4 \times \frac{1}{2} \left(\left| \frac{\eta(\sigma)}{\vartheta_4(\sigma)} \right|^2 + \left| \frac{\eta(\sigma)}{\vartheta_3(\sigma)} \right|^2 \right) Z_{\mathrm{Dirac}}. \end{aligned} \tag{30}$$

Then the complete modular invariant $\mathbb{Z}_2$ orbifold partition function has the form

$$Z_{\mathbb{Z}_2-\mathrm{orb}} = \frac{1}{2} \left(Z(\tau, \rho) + 4 \left| \frac{\eta(\sigma)}{\vartheta_2(\sigma)} \right|^2 + 4 \left| \frac{\eta(\sigma)}{\vartheta_3(\sigma)} \right|^2 + 4 \left| \frac{\eta(\sigma)}{\vartheta_4(\sigma)} \right|^2 \right) Z_{\mathrm{Dirac}}. \tag{31}$$

The (c, c), (a, c), and their complex conjugates, R ground states as well as the Witten index for the $\mathbb{Z}_2$ orbifold are the same as those for the $N = 2$ theories with $c = 3$.

4. The $\mathbb{Z}_3$ orbifold

By dividing the $\mathbb{Z}_3$ symmetry from (15) for $\tau = \mathrm{e}^{2\pi \mathrm{i}/3}$ and arbitrary ρ, we may construct the $\mathbb{Z}_3$ orbifold model. The action of $g \in \mathbb{Z}_3$ on the bosonic Hilbert space sectors is given by

$$g|m_1, m_2, n_1, n_2\rangle = |m_2, -m_1 - m_2, n_2 - n_1, -n_1\rangle. \tag{32}$$

By (22), the untwisted sector partition function is

$$Z_u = (q\bar{q})^{-\frac{1}{8}} \operatorname{tr}_{\tilde{\mathcal{H}}_u} \tfrac{1}{3}(1 + g + g^2) q^{L_0} \bar{q}^{\bar{L}_0} y^{J_0} \bar{y}^{\bar{J}_0}.$$

By taking into account the equations (16), (17), (20), (21) and (32), the first term in the trace is equal to the original partition function (15); the second and third terms only receive a contribution from the Hilbert space sector built on $|0, 0, 0, 0\rangle$. The untwisted sector partition function is therefore given by

$$Z_u = \frac{1}{3} \left(Z(\tau = \mathrm{e}^{2\pi \mathrm{i}/3}, \rho, z) + \frac{3}{2} \sum_{i=1}^{4} \left(\left| \frac{\vartheta_i(z + \frac{1}{3}, \sigma)}{\vartheta_1(\frac{1}{3}, \sigma)} \right|^2 + \left| \frac{\vartheta_i(z - \frac{1}{3}, \sigma)}{\vartheta_1(\frac{1}{3}, \sigma)} \right|^2 \right) \right).$$

$\mathbb{Z}_3$ does not act freely on the hexagonal torus. Thus one must consider new sectors, the twisted ones. In the $T^2/\mathbb{Z}_3$ ($\tau = \mathrm{e}^{2\pi \mathrm{i}/3}$) manifold, there are three fixed points, and one can obtain three Hilbert space sectors corresponding to the expansion of the field about each of these fixed points. However, these three sectors give the same physics. The conformal weight of the bosonic twisted ground state is $(\frac{1}{9}, \frac{1}{9})$. For a fermion, the T sector conformal weight is $(\frac{1}{18}, \frac{1}{18})$. Thus the total conformal weight of the T sector is then $(\frac{1}{6}, \frac{1}{6})$. States in the T sector are generated by the action of creation operators on the twisted ground state.

By considering the equations (16), (17), (25), (26) and (32), the T sector partition function may be written as

$$\begin{aligned} Z_t &= (q\bar{q})^{-\frac{1}{8}} \operatorname{tr}_{\tilde{\mathcal{H}}_t} \tfrac{1}{3}(1 + g + g^2) q^{L_0} \bar{q}^{\bar{L}_0} y^{J_0} \bar{y}^{\bar{J}_0} \\ &= 3 \times \frac{1}{2 \times 3} \sum_{i=1}^{4} \sum_{l=-1}^{1} \left(\left| y^{-\frac{1}{3}} \frac{\vartheta_i(z + \frac{l}{3} - \frac{\sigma}{3}, \sigma)}{\vartheta_1(\frac{l}{3} - \frac{\sigma}{3}, \sigma)} \right|^2 + \left| y^{\frac{1}{3}} \frac{\vartheta_i(z + \frac{l}{3} + \frac{\sigma}{3}, \sigma)}{\vartheta_1(\frac{l}{3} + \frac{\sigma}{3}, \sigma)} \right|^2 \right). \end{aligned} \tag{33}$$

Then the complete modular invariant $\mathbb{Z}_3$ orbifold partition function is

$$Z_{\mathbb{Z}_3-\mathrm{orb}} = \frac{1}{3}\Bigg(Z(\tau = \mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}, \rho, z) + \frac{3}{2}\sum_{i=1}^{4}\sum_{s=1}^{2}\left|\frac{\vartheta_i(z+\frac{s}{3},\sigma)}{\vartheta_1(\frac{s}{3},\sigma)}\right|^2 + \frac{3}{2}\sum_{l=-1}^{1}\sum_{i=1}^{4}\left(\left|y^{-\frac{1}{3}}\frac{\vartheta_i(z+\frac{l}{3}-\frac{\sigma}{3},\sigma)}{\vartheta_1(\frac{l}{3}-\frac{\sigma}{3},\sigma)}\right|^2 + \left|y^{\frac{1}{3}}\frac{\vartheta_i(z+\frac{l}{3}+\frac{\sigma}{3}\sigma)}{\vartheta_1(\frac{l}{3}+\frac{\sigma}{3},\sigma)}\right|^2\right)\Bigg). \tag{34}$$

We find eight R ground states with conformal dimension $(h,\bar{h}) = (\frac{1}{8},\frac{1}{8})$ and with charges $(\pm\frac{1}{2},\pm\frac{1}{2})$, $3\times(\pm\frac{1}{6},\pm\frac{1}{6})$, eight NS chiral primary states with conformal dimensions $(0,0)$, $(\frac{1}{2},\frac{1}{2})$, $3\times(\frac{1}{6},\frac{1}{6})$, $3\times(\frac{1}{3},\frac{1}{3})$ and with charges $(0,0)$, $(1,1)$, $3\times(\frac{1}{3},\frac{1}{3})$, $3\times(\frac{2}{3},\frac{2}{3})$, as well as eight NS antichiral primary states having the same conformal dimensions but the opposite charges to the NS chiral fields. By (5) with $\eta=\frac{1}{2}$, the ground states of the R sector flow to the (c,c) primary states of the NS sector, namely

$$\begin{array}{l} \text{R ground states} \longleftrightarrow \text{NS chiral states} \\ q^{\frac{1}{8}}\bar{q}^{\frac{1}{8}}y^{-\frac{1}{2}}\bar{y}^{-\frac{1}{2}} \longleftrightarrow 1 \\ q^{\frac{1}{8}}\bar{q}^{\frac{1}{8}}y^{\frac{1}{2}}\bar{y}^{\frac{1}{2}} \longleftrightarrow q^{\frac{1}{2}}\bar{q}^{\frac{1}{2}}y\bar{y} \\ 3\times q^{\frac{1}{8}}\bar{q}^{\frac{1}{8}}y^{-\frac{1}{6}}\bar{y}^{-\frac{1}{6}} \longleftrightarrow 3\times q^{\frac{1}{6}}\bar{q}^{\frac{1}{6}}y^{\frac{1}{3}}\bar{y}^{\frac{1}{3}} \\ 3\times q^{\frac{1}{8}}\bar{q}^{\frac{1}{8}}y^{\frac{1}{6}}\bar{y}^{\frac{1}{6}} \longleftrightarrow 3\times q^{\frac{1}{3}}\bar{q}^{\frac{1}{3}}y^{\frac{2}{3}}\bar{y}^{\frac{2}{3}}. \end{array} \tag{35}$$

(Here $q=\mathrm{e}^{2\pi\mathrm{i}\sigma}$ and $y=\mathrm{e}^{2\pi\mathrm{i}z}$.) If we reverse the direction of the spectral flow, we obtain an isomorphism between the (a,a) primary states and the ground states of the R sector. By (6), (7) and (35) the Witten index and the Poincaré polynomial for the (c,c) states are

$$\mathrm{Tr}(-1)^F = 8 \qquad P(t,\bar{t})_{(c,c)} = 1 + t\bar{t} + 3t^{\frac{1}{3}}\bar{t}^{\frac{1}{3}} + 3t^{\frac{2}{3}}\bar{t}^{\frac{2}{3}}. \tag{36}$$

The spectral flow from the NS sector to the NS sector can be obtained by the flow parameter $\eta=1$. In the spectrum, there are no nontrivial (a,c) or its conjugate (c,a) states.

5. The $\mathbb{Z}_4$ orbifold

In this section, by dividing the $\mathbb{Z}_4$ symmetry from (15) for $\tau=\mathrm{i}$ and arbitrary ρ, we may formulate the $\mathbb{Z}_4$ orbifold model. The action of $g\in\mathbb{Z}_4$ on the bosonic ground state sectors is given by

$$g|m_1,m_2,n_1,n_2\rangle = |m_2,-m_1,n_2,-n_1\rangle. \tag{37}$$

Under the rotation group $\mathbb{Z}_4$ the square lattice has three fixed points. An analysis similar to the $\mathbb{Z}_3$ orbifold shows there are T sectors associated with those fixed points, namely one fixed point corresponding to the $\mathbb{Z}_2$ twist and two for the $\mathbb{Z}_4$ twist. The weights of the bosonic and fermionic $\mathbb{Z}_4$ twisted ground state are $(\frac{3}{32},\frac{3}{32})$ and $(\frac{1}{32},\frac{1}{32})$, respectively. Thus the total conformal weight of the $\mathbb{Z}_4$ T sector is $(\frac{1}{8},\frac{1}{8})$. The total $\mathbb{Z}_4$ orbifold partition function can be obtained by summing over untwisted, $\mathbb{Z}_2$ and $\mathbb{Z}_4$ T sector partition functions:

$$Z_{\mathbb{Z}_4-\mathrm{orb}}(\tau=i,\rho,z) = Z_u + Z_{2t} + Z_{4t}.$$

By (16), (17), (20)–(22) and (37), we obtain the following untwisted sector partition function:

$$\begin{aligned} Z_u &= (q\bar{q})^{-\frac{1}{8}}\,\mathrm{tr}_{\tilde{\mathcal{H}}_u}\,\tfrac{1}{4}(1+g+g^2+g^3)q^{L_0}\bar{q}^{\bar{L}_0}y^{J_0}\bar{y}^{\bar{J}_0} \\ &= \frac{1}{4}\left(Z(\tau=\mathrm{i},\rho,z) + \sum_{i=1}^{4}\left|\frac{\vartheta_i(z,\sigma)}{\vartheta_2(\sigma)}\right|^2 + \sum_{i=1}^{4}\sum_{s=1}^{3}\left|\frac{\vartheta_i(z+\frac{s}{4},\sigma)}{\vartheta_1(\frac{s}{4},\sigma)}\right|^2\right). \end{aligned}$$

By (16), (17), (25)–(27) and (37), $\mathbb{Z}_4$ the T sector partition function may have the form

$$Z_{4t} = \frac{1}{4}\sum_{i,l=1}^{4}\left(\left|y^{-\frac{1}{4}}\frac{\vartheta_i(z+\frac{l}{4}-\frac{\sigma}{4},\sigma)}{\vartheta_1(\frac{l}{4}-\frac{\sigma}{4},\sigma)}\right|^2+\left|y^{\frac{1}{4}}\frac{\vartheta_i(z+\frac{l}{4}+\frac{\sigma}{4},\sigma)}{\vartheta_1(\frac{l}{4}+\frac{\sigma}{4},\sigma)}\right|^2+\left|\frac{\vartheta_i(z+\frac{l}{4},\sigma)}{\vartheta_4(\frac{l}{4},\sigma)}\right|^2\right).$$

The $\mathbb{Z}_2$ T sector partition function can be read off from (30) by omitting the factor of four. Thus, we may write the modular invariant $\mathbb{Z}_4$ orbifold partition function in the following form:

$$\begin{aligned}Z_{\mathbb{Z}_4-\mathrm{orb}} = \frac{1}{4}\sum_{i,l=1}^{4}\Bigg(&Z(\tau=\mathrm{I},\rho,z)+\sum_{j=2}^{4}\left|\frac{\vartheta_i(z,\sigma)}{\vartheta_j(\sigma)}\right|^2+\sum_{s=1}^{3}\left|\frac{\vartheta_i(z+\frac{s}{4},\sigma)}{\vartheta_1(\frac{s}{4},\sigma)}\right|^2\\ &+\left|\frac{\vartheta_i(z+\frac{l}{4},\sigma)}{\vartheta_4(\frac{l}{4},\sigma)}\right|^2+\left|y^{-\frac{1}{4}}\frac{\vartheta_i(z+\frac{l}{4}-\frac{\sigma}{4},\sigma)}{\vartheta_1(\frac{l}{4}-\frac{\sigma}{4},\sigma)}\right|^2+\left|y^{\frac{1}{4}}\frac{\vartheta_i(z+\frac{l}{4}+\frac{\sigma}{4},\sigma)}{\vartheta_1(\frac{l}{4}+\frac{\sigma}{4},\sigma)}\right|^2\Bigg).\end{aligned} \tag{38}$$

In the spectrum there are nine R ground states which flow to the NS chiral states under the spectral flow operation (5)with flow parameter $\eta = \frac{1}{2}$

$$\begin{aligned}&\text{R ground states} \longleftrightarrow \text{NS chiral states}\\ &q^{\frac{1}{8}}\bar{q}^{\frac{1}{8}}y^{-\frac{1}{2}}\bar{y}^{-\frac{1}{2}} \longleftrightarrow 1\\ &q^{\frac{1}{8}}\bar{q}^{\frac{1}{8}}y^{\frac{1}{2}}\bar{y}^{\frac{1}{2}} \longleftrightarrow q^{\frac{1}{2}}\bar{q}^{\frac{1}{2}}y\bar{y}\\ &2\times q^{\frac{1}{8}}\bar{q}^{\frac{1}{8}}y^{-\frac{1}{4}}\bar{y}^{-\frac{1}{4}} \longleftrightarrow 2\times q^{\frac{1}{8}}\bar{q}^{\frac{1}{8}}y^{\frac{1}{4}}\bar{y}^{\frac{1}{4}}\\ &2\times q^{\frac{1}{8}}\bar{q}^{\frac{1}{8}}y^{\frac{1}{4}}\bar{y}^{\frac{1}{4}} \longleftrightarrow 2\times q^{\frac{3}{8}}\bar{q}^{\frac{3}{8}}y^{\frac{3}{4}}\bar{y}^{\frac{3}{4}}\\ &3\times q^{\frac{1}{8}}\bar{q}^{\frac{1}{8}} \longleftrightarrow 3\times q^{\frac{1}{4}}\bar{q}^{\frac{1}{4}}y^{\frac{1}{2}}\bar{y}^{\frac{1}{2}}.\end{aligned} \tag{39}$$

There are nine (a, a) states which are given by the complex conjugation of (c, c) states. As in the $\mathbb{Z}_3$ orbifold case, one can obtain isomorphism between the (a, a) primary states and the ground states of the R sector by reversing the direction of the spectral flow. By (6), (7) and (39), the Witten index and the Poincaré polynomial for the (c, c) states are

$$\begin{aligned}&\mathrm{Tr}(-1)^F = 9\\ &P(t,\bar{t})_{(c,c)} = 1+t\bar{t}+3t^{\frac{1}{2}}\bar{t}^{\frac{1}{2}}+2t^{\frac{1}{4}}\bar{t}^{\frac{1}{4}}+2t^{\frac{3}{4}}\bar{t}^{\frac{3}{4}}.\end{aligned} \tag{40}$$

With the spectral flow parameter $\eta = 1$, the NS sector returns to the NS sector. One notes that the $\mathbb{Z}_4$ orbifold model contains only (c, c) and their conjugate (a, a) states. For this model, the (a, c) and (c, a) states are trivial and consist only of the vacumm state.

6. The $\mathbb{Z}_6$ orbifold

We now construct a $\mathbb{Z}_6$ orbifold model by dividing $\mathbb{Z}_6$ symmetry from (15) for $\tau = \mathrm{e}^{2\pi\frac{\mathrm{i}}{3}}$ and arbitrary ρ. The bosonic ground state sectors transform as follows under the action of $g \in \mathbb{Z}_6$:

$$g|m_1, m_2, n_1, n_2\rangle = |m_1 + m_2, -m_1, n_2, -n_1 + n_2\rangle. \tag{41}$$

The hexagonal torus has three fixed points under the $\mathbb{Z}_6$ rotation symmetry. There is a T sector associated with each of them. These are $\mathbb{Z}_2$, $\mathbb{Z}_3$ and $\mathbb{Z}_6$ T sectors. The conformal dimensions of the bosonic and fermionic $\mathbb{Z}_6$ twisted ground state are $(\frac{5}{72}, \frac{5}{72})$ and $(\frac{1}{72}, \frac{1}{72})$, respectively. Thus the total conformal weight of the $\mathbb{Z}_6$ twisted ground state is $(\frac{1}{12}, \frac{1}{12})$. The $\mathbb{Z}_6$ orbifold partition function is the sum of partition functions of the untwisted, $\mathbb{Z}_2$, $\mathbb{Z}_3$ and $\mathbb{Z}_6$ T sectors:

$$Z_{\mathbb{Z}_6-\mathrm{orb}}(\tau = \mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}, \rho, z) = Z_u + Z_{2t} + Z_{3t} + Z_{6t}.$$

By applying the same method as for the construction of the $\mathbb{Z}_2$, $\mathbb{Z}_3$ and $\mathbb{Z}_4$ orbifolds, we obtain the following untwisted $\mathbb{Z}_6$ orbifold partition function:

$$\begin{aligned}
Z_u &= (q\bar{q})^{-\frac{1}{8}} \operatorname{tr}_{\tilde{\mathcal{H}}_u} \tfrac{1}{6}(1+g+\cdots+g^5) q^{L_0} \bar{q}^{\bar{L}_0} y^{J_0} \bar{y}^{\bar{J}_0} \\
&= \frac{1}{6}\Bigg(Z(\tau = \mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}, \rho, z) + \frac{3}{2}\sum_{i=1}^{4} \left| \frac{\vartheta_i(z,\sigma)}{\vartheta_2(\sigma)} \right|^2 \\
&\quad + \frac{3}{2}\sum_{i=1}^{4}\sum_{s=1}^{2} \left| \frac{\vartheta_i(z+\frac{s}{3},\sigma)}{\vartheta_1(\frac{s}{3},\sigma)} \right|^2 + \frac{1}{2}\sum_{i=1}^{4}\sum_{l=-1}^{1} \left| \frac{\vartheta_i(z+\frac{l}{3},\sigma)}{\vartheta_2(\frac{l}{3},\sigma)} \right|^2 \Bigg).
\end{aligned}$$

The $\mathbb{Z}_2$ and $\mathbb{Z}_3$ T sector partition functions can be read off from (30) and (33) by omitting the factor of four and three, respectively. The $\mathbb{Z}_6$ T sector partition function may have the form

$$\begin{aligned}
Z_{6t} &= \frac{1}{12}\sum_{i,k=1}^{4}\sum_{l=-1}^{1} \Bigg(\left| \frac{\vartheta_i(z+\frac{l}{3},\sigma)}{\vartheta_3(\frac{l}{3},\sigma)} \right|^2 + \left| \frac{\vartheta_i(z+\frac{l}{3},\sigma)}{\vartheta_4(\frac{l}{3},\sigma)} \right|^2 \\
&\quad + \left| y^{\frac{1}{3}} \frac{\vartheta_i(z+\frac{l}{3}+\frac{\sigma}{3},\sigma)}{\vartheta_k(\frac{l}{3}+\frac{\sigma}{3},\sigma)} \right|^2 + \left| y^{-\frac{1}{3}} \frac{\vartheta_i(z+\frac{l}{3}-\frac{\sigma}{3},\sigma)}{\vartheta_k(\frac{l}{3}-\frac{\sigma}{3},\sigma)} \right|^2 \Bigg).
\end{aligned}$$

All in all we obtain the following modular invariant partition function

$$\begin{aligned}
Z_{\mathbb{Z}_6-\mathrm{orb}} &= \frac{1}{6}\sum_{i=1}^{4}\sum_{j=2}^{4} \Bigg(Z(\tau = \mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}, \rho, z) + \frac{3}{2}\left| \frac{\vartheta_i(z,\sigma)}{\vartheta_j(\sigma)} \right|^2 + \frac{3}{2}\sum_{s=1}^{2} \left| \frac{\vartheta_i(z+\frac{s}{3},\sigma)}{\vartheta_1(\frac{s}{3},\sigma)} \right|^2 \\
&\quad + \frac{1}{2}\sum_{l=-1}^{1} \Bigg(2\left| y^{-\frac{1}{3}} \frac{\vartheta_i(z+\frac{l}{3}-\frac{\sigma}{3},\sigma)}{\vartheta_1(\frac{l}{3}-\frac{\sigma}{3},\sigma)} \right|^2 + 2\left| y^{\frac{1}{3}} \frac{\vartheta_i(z+\frac{l}{3}+\frac{\sigma}{3},\sigma)}{\vartheta_1(\frac{l}{3}+\frac{\sigma}{3},\sigma)} \right|^2 \\
&\quad + \left| \frac{\vartheta_i(z+\frac{l}{3},\sigma)}{\vartheta_j(\frac{l}{3},\sigma)} \right|^2 + \left| y^{-\frac{1}{3}} \frac{\vartheta_i(z+\frac{l}{3}-\frac{\sigma}{3},\sigma)}{\vartheta_j(\frac{l}{3}-\frac{\sigma}{3},\sigma)} \right|^2 + \left| y^{\frac{1}{3}} \frac{\vartheta_i(z+\frac{l}{3}+\frac{\sigma}{3},\sigma)}{\vartheta_j(\frac{l}{3}+\frac{\sigma}{3},\sigma)} \right|^2 \Bigg)\Bigg).
\end{aligned} \tag{42}$$

In this model there are ten R ground states. Again we connect the ground states of the R sector with NS chiral primary states using equation (5) with $\eta = \frac{1}{2}$.

$$\begin{aligned}
&\text{R ground states} \longleftrightarrow \text{NS chiral states} \\
&q^{\frac{1}{8}}\bar{q}^{\frac{1}{8}}y^{-\frac{1}{2}}\bar{y}^{-\frac{1}{2}} \longleftrightarrow 1 \\
&q^{\frac{1}{8}}\bar{q}^{\frac{1}{8}}y^{\frac{1}{2}}\bar{y}^{\frac{1}{2}} \longleftrightarrow q^{\frac{1}{2}}\bar{q}^{\frac{1}{2}}y\bar{y} \\
&2\times q^{\frac{1}{8}}\bar{q}^{\frac{1}{8}} \longleftrightarrow 2\times q^{\frac{1}{4}}\bar{q}^{\frac{1}{4}}y^{\frac{1}{2}}\bar{y}^{\frac{1}{2}} \\
&q^{\frac{1}{8}}\bar{q}^{\frac{1}{8}}y^{\frac{1}{3}}\bar{y}^{\frac{1}{3}} \longleftrightarrow q^{\frac{5}{12}}\bar{q}^{\frac{5}{12}}y^{\frac{5}{6}}\bar{y}^{\frac{5}{6}} \\
&q^{\frac{1}{8}}\bar{q}^{\frac{1}{8}}y^{-\frac{1}{3}}\bar{y}^{-\frac{1}{3}} \longleftrightarrow q^{\frac{1}{12}}\bar{q}^{\frac{1}{12}}y^{\frac{1}{6}}\bar{y}^{\frac{1}{6}} \\
&2\times q^{\frac{1}{8}}\bar{q}^{\frac{1}{8}}y^{\frac{1}{6}}\bar{y}^{\frac{1}{6}} \longleftrightarrow 2\times q^{\frac{1}{3}}\bar{q}^{\frac{1}{3}}y^{\frac{2}{3}}\bar{y}^{\frac{2}{3}} \\
&2\times q^{\frac{1}{8}}\bar{q}^{\frac{1}{8}}y^{-\frac{1}{6}}\bar{y}^{-\frac{1}{6}} \longleftrightarrow 2\times q^{\frac{1}{6}}\bar{q}^{\frac{1}{6}}y^{\frac{1}{3}}\bar{y}^{\frac{1}{3}}.
\end{aligned} \tag{43}$$

By (6), (7) and (43), the Witten index and the Poincaré polynomials for the (c, c) states are

$$\begin{aligned}
&\mathrm{Tr}(-1)^F = 10 \\
&P(t,\bar{t})_{(c,c)} = 1 + t\bar{t} + 2t^{\frac{1}{2}}\bar{t}^{\frac{1}{2}} + t^{\frac{5}{6}}\bar{t}^{\frac{5}{6}} + t^{\frac{1}{6}}\bar{t}^{\frac{1}{6}} + 2t^{\frac{2}{3}}\bar{t}^{\frac{2}{3}} + 2t^{\frac{1}{3}}\bar{t}^{\frac{1}{3}}.
\end{aligned} \tag{44}$$

The (a, a) states are given by the complex conjugation of (c, c) states. We found no (a, c) or (c, a) states in this model.

7. The $N = 2$ Landau–Ginzburg model

In this section, we first review some of the facts of the $N = 2$ superconformal Landau–Ginzburg theories by following the articles [11, 16], then we check the spectrum of the (c, c) fields and the Witten index. The $N = 2$ superconformal Landau–Ginzburg action takes the following form:

$$S = \int \mathrm{d}^2 z\, \mathrm{d}^4\theta\, K(\Phi_i, \bar{\Phi}_i) + \left(\int \mathrm{d}^2 z\, \mathrm{d}^2\theta\, W(\Phi_i) + \text{HC} \right). \tag{45}$$

$\Phi_i (i = 1, 2, \dots, n)$ are the $N = 2n$ chiral scalar superfields which satisfy the condition $\bar{D}_\pm \Phi_i = D_\pm \bar{\Phi}_i = 0$, where the superderivative is defined as $D_\pm = \frac{\partial}{\partial\theta^\pm} + \theta^\mp \frac{\partial}{\partial z}$. The first term (K) is called the Kähler potential. It includes derivatives of the superfields. The conformal dimension of these fields is greater than (1, 1). Such fields are called irrelevant. The second term (W) is called the superpotential and is a holomorphic function of the superfields. It contains only relevant fields, i.e. fields with conformal dimension (1, 1) or less than (1, 1). The holomorphic superpotential $W(\Phi_i)$ is a quasi-homogeneous function with isolated singularities at $\Phi_i = 0$. In other words $W(\Phi_i)$ is called quasi-homogeneous if it satisfies

$$W(\lambda^{w_i} \Phi_i) = \lambda^d W(\Phi_i) \qquad \text{for} \quad \Phi_i \to \lambda^{w_i} \Phi_i \tag{46}$$

where w^i and d are integers with no common factors. It has isolated singularity at $\Phi_i = 0$ if it satisfies

$$W(\Phi_i)|_0 = 0 \qquad \partial_i W(\Phi_j)|_0 = 0.$$

For every isolated quasi-homogeneous superpotential, there exists an $N = 2$ superconformal field theory. One can read off the $U(1)$ charge of the lowest component of the chiral superfields Φ_i from the action (45). The θ integrals in the first term have (*left, right*) charges $(-1, -1)$. Because of neutrality of the action $W(\Phi_i)$ has charge (1, 1). Thus, the chiral superfield Φ_i must have charge $q_i = w_i/d$ for both its left-right-moving components. Now one notes that for any state in the Landau–Ginzburg theory $q_L - q_R$ is always an integer. This is true for the chiral superfield Φ_i, as it has equal left–right charges. Moreover, it is also true for the most general fields because they are obtained by taking products of Φ_i and $\bar{\Phi}_i$, as well as products of their superderivatives. This implies that one can apply spectral flow to the Landau–Ginzburg models.

The local *ring* $\mathcal{R}$ of the superpotential $W(\Phi_i)$ of the Landau–Ginzburg model is obtained by taking into account all monomials of chiral superfields Φ_i and setting $\partial_i W(\Phi_j)|_0 = 0$. The number of elements of the ring is denoted by $\mu = \dim \mathcal{R}$. It is called the multiplicity of $W(\Phi_i)$. It is also equal to the Witten index $\mathrm{Tr}(-1)^F$.

The *modality* (or moduli) is the number of free parameters in the theory. m of a quasi-homogeneous superpotential with isolated singularities is given by the number of chiral primary states with charge greater than or equal to one.

The *Poincaré polynomial* [11] for the Landau–Ginzburg theories is

$$P(t) = \mathrm{Tr}_{\mathcal{R}}\, t^{dJ_0} = \prod_{i=1}^{n} \frac{1 - t^{d-w_i}}{1 - t^{w_i}} \qquad \text{or} \qquad P(t, \bar{t}) = \mathrm{Tr}_{\mathcal{R}}\, t^{J_0} \bar{t}^{\bar{J}_0}. \tag{47}$$

This polynomial is only a function of $t\bar{t}$ (because Landau–Ginzburg primary chiral fields have equal left–right charges). For convenience, $t\bar{t}$ is replaced by the variable t^d, where d is defined in (46). The *Witten index* [11] is

$$\mathrm{Tr}(-1)^F = P(t = 1) = \mu = \prod_{i=1}^{n} \frac{d - w_i}{w_i} = \prod_{i=1}^{n} \left(\frac{1}{q_i} - 1 \right). \tag{48}$$

The highest charge and conformal dimension of the chiral primary state $|\chi\rangle$ [11] are given as

$$q_\chi = \sum_{i=1}^{\infty} \frac{d - 2w_i}{d} = \sum_i (1 - 2q_i) \qquad h_\chi = \frac{q_\chi}{2} = \sum_{i=1}^{n} \left(\frac{1}{2} - q_i\right).$$

By using $h_\chi = \frac{c}{6}$, the central charge of the Landau–Ginzburg theory is given as

$$c = 6h_\chi = 6\sum_{i=1}^{n} (\tfrac{1}{2} - q_i).$$

It is well known [16] that the *quasi-homogeneous superpotentials* with isolated singularities for modality $m = 1$ of the Landau–Ginzburg theories at $c = 3$ are equivalent to the $\mathbb{Z}_M$, $M \in \{3, 4, 6\}$, orbifolds of the $N = 2$ theories at $c = 3$. The corresponding superpotentials are given as

$$W_3(\Phi_1, \Phi_2, \Phi_3) = \Phi_1^3 + \Phi_2^3 + \Phi_3^3 + 6a\Phi_1\Phi_2\Phi_3 \qquad a^3 + 27 \neq 0 \tag{49}$$

$$W_4(\Phi_1, \Phi_2) = \Phi_1^4 + \Phi_2^4 + a\Phi_1^2\Phi_2^2 \qquad a^2 \neq 4 \tag{50}$$

$$W_6(\Phi_1, \Phi_2) = \Phi_1^3 + \Phi_2^6 + a\Phi_1^2\Phi_2^2 \qquad 4a^3 + 27 \neq 0. \tag{51}$$

With the knowledge in this section, we may write the basic linearly independent elements of the (c, c) ring of superpotential (49) in the following form:

Chiral fields	1	Φ_1	Φ_2	Φ_3	$\Phi_1\Phi_2$	$\Phi_1\Phi_3$	$\Phi_2\Phi_3$	$\Phi_1\Phi_2\Phi_3$
Charges	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	1
Dimensions	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}$.

(52)

By (47), (48) and (52), the Witten index and Poincaré polynomial are

$$\mathrm{Tr}(-1)^F = 8 \qquad P(t, \bar{t})_{(c,c)} = \mathrm{Tr}_{\mathcal{R}}\, t^{J_0}\bar{t}^{\bar{J}_0} = 1 + t\bar{t} + 3t^{\frac{1}{3}}\bar{t}^{\frac{1}{3}} + 3t^{\frac{2}{3}}\bar{t}^{\frac{2}{3}}. \tag{53}$$

For the superpotential (50) we have

Chiral fields	1	Φ_1	Φ_2	$\Phi_1\Phi_2$	Φ_1^2	Φ_2^2	$\Phi_1^2\Phi_2$	$\Phi_1\Phi_2^2$	$\Phi_1^2\Phi_2^2$
Charges	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{4}$	1
Dimensions	0	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{2}$.

(54)

By (47), (48) and (54), the Witten index and Poincaré polynomial are

$$\mathrm{Tr}(-1)^F = 9 \qquad P(t, \bar{t})_{(c,c)} = 1 + t\bar{t} + 3t^{\frac{1}{2}}\bar{t}^{\frac{1}{2}} + 2t^{\frac{1}{4}}\bar{t}^{\frac{1}{4}} + 2t^{\frac{3}{4}}\bar{t}^{\frac{3}{4}}. \tag{55}$$

Similarly, for the superpotential (51) we may obtain

Chiral fields	1	Φ_1	Φ_2	$\Phi_1\Phi_2$	Φ_2^2	Φ_2^3	Φ_2^4	$\Phi_1\Phi_2^2$	$\Phi_1\Phi_2^3$	$\Phi_1\Phi_2^4$
Charges	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	1
Dimensions	0	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{5}{12}$	$\frac{1}{2}$.

(56)

By (47), (48) and (56), the Witten index and Poincaré polynomial are

$$\mathrm{Tr}(-1)^F = 10 \qquad P(t, \bar{t})_{(c,c)} = 1 + t\bar{t} + 2t^{\frac{1}{2}}\bar{t}^{\frac{1}{2}} + t^{\frac{5}{6}}\bar{t}^{\frac{5}{6}} + t^{\frac{1}{6}}\bar{t}^{\frac{1}{6}} + 2t^{\frac{2}{3}}\bar{t}^{\frac{2}{3}} + 2t^{\frac{1}{3}}\bar{t}^{\frac{1}{3}}. \tag{57}$$

Conclusion

The partition functions for $\mathbb{Z}_M$ orbifolds have been calculated. The Witten indices, the spectrum of (chiral, chiral) fields for the $\mathbb{Z}_M$, $M \in \{3, 4, 6\}$, orbifolds and for the Landau–Ginzburg superpotentials (49–51) are given in equations (36), (40), (44), (35), (39), (43) and (52), (54), (56), (53), (55), (57), respectively. The results are in in agreement with the Landau–Ginzburg predictions of Vafa and Warner.

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